Last class: $\Phi : G \to H$ is a homomorphism

if $\Phi(ab) = \Phi(a)\Phi(b)$ for all $a, b \in G$

Φ need not be 1-1 or onto

$\text{Ker } \Phi = \{ g \in G, \ \Phi(g) = e_H \}$

identity of H

<u>Theorem</u> $\text{Ker } \Phi$ is a normal subgroup of G

Proof already showed in last lecture: $\text{Ker } \Phi$ is a subgroup

to show that it is normal, use normal subgroup test

i.e. need to show for any $k \in \text{Ker } \Phi$ and any $g \in G$

$$g k g^{-1} \in \ker \Phi \ !$$

$$\Phi(g k g^{-1}) = \Phi(g)\Phi(k)\Phi(g)^{-1} = \Phi(g) e_H \Phi(g)^{-1} = \Phi(g)\Phi(g)^{-1} = e_H$$

$\Rightarrow \; ghg^{-1} \in \operatorname{Ker} \Phi$ ✓

Example: ① $\Phi : S_n \to \mathbb{Z}_2$

$$\Phi(\pi) = \begin{cases} 1 & \pi \text{ odd permutation} \\ 0 & \pi \text{ even } \text{"} \end{cases}$$

$\operatorname{Ker} \Phi = \{\pi, \; \Phi(\pi) = 0\} = \{\pi, \; \pi \text{ even}\} = A_n$

have already checked before: $A_n \trianglelefteq S_n$.

② $\Phi : \mathbb{C}^* \to \mathbb{C}^*$ (recall: $\mathbb{C}^* = (\mathbb{C} \setminus \{0\}, \cdot)$)

$\Phi(z) = z^4$

hom: $\Phi(zy) = (zy)^4 \underset{\uparrow \text{ abelian}}{=} z^4 y^4 = \Phi(z)\Phi(y)$ ✓

$\operatorname{Ker} \Phi = \{z, \; z^4 = 1\} = \{\pm 1, \pm i\}$.

<u>Corollary</u> $\Phi(a) = \Phi(b) \Rightarrow a \operatorname{Ker}\Phi = b \operatorname{Ker}\Phi$
(same cosets)

Proof. $\Phi(a) = \Phi(b) \quad | \ \Phi(a)^{-1} \cdot$

$\Rightarrow \quad e_H = \Phi(a)^{-1}\Phi(b)$

$= \Phi(a^{-1}b)$ (hom. property)

$\Rightarrow \quad a^{-1}b \in \operatorname{Ker}\Phi \quad | \ a \cdot$

$b \in a \operatorname{Ker}\Phi$

$\Rightarrow \quad b \operatorname{Ker}\Phi = a \operatorname{Ker}\Phi$

✓

Application: Find all solutions of $z^4 = 2$

Solution: Assume z_1 and z_2 are two such solution

$\Rightarrow \quad z_1^4 = z_2^4$

$\Rightarrow \quad \Phi(z_1) = \Phi(z_2)$ where $\Phi: z \to z^4$

$\Rightarrow \quad z_2 \operatorname{Ker} \Phi = z_1 \operatorname{Ker} \Phi$

$\Leftarrow \quad z_2 \in z_1 \operatorname{Ker} \Phi = z_1 \{\pm 1, \pm i\} = \{\pm \sqrt[4]{2}, \pm i \sqrt[4]{2}\}$

for known solution $z_1 = \sqrt[4]{2}$

Result: $\{\pm \sqrt[4]{2}, \pm i \sqrt[4]{2}\}$ are all solutions

Recall: $|S_n/A_n| = 2$

$$\Rightarrow \quad S_n/A_n \cong \mathbb{Z}_2 \qquad \left(\text{any group of order } 2 \cong \mathbb{Z}_2\right)$$

on the other hand: we have hom.

$$\Phi : S_n \to \mathbb{Z}_2$$

whose kernel is A_n

$$\Rightarrow \quad S_n/\ker \Phi \cong \mathbb{Z}_2 = \Phi(S_n)$$

No coincidence!

First Isomorphism Theorem

$\Phi : G \to H$ homomorphism

Then $G/\ker\Phi \cong \Phi(G) = \{\Phi(g), g \in G\}$

Proof need to find an isomorphism. $\psi : G/\ker\Phi \to \Phi(G)$

Define: $\psi(g\ker\Phi) = \Phi(g)$

- well-defined!

  assume $g\ker\Phi = \tilde{g}\ker\Phi$

  $\Rightarrow \tilde{g} = gk$ for some $k \in \ker\Phi$

  $\Rightarrow \Phi(\tilde{g}) = \Phi(gk) \overset{\text{homom.}}{=} \Phi(g)\underbrace{\Phi(k)}_{e_H} = \Phi(g)e_H = \Phi(g)$

  $\Rightarrow \Phi$ well-defined.

onto: for given $h \in \Phi(G)$
we have g s.t. $h = \Phi(g)$
$$\Rightarrow \psi(g \operatorname{Ker} \Phi) = \Phi(g) = h$$ ✓

1-1: assume $\psi(g \operatorname{Ker} \Phi) = \psi(\tilde{g} \operatorname{Ker} \Phi)$
$$\Rightarrow \Phi(g) = \Phi(\tilde{g})$$
by Corollary earlier this lecture
$$\Rightarrow g \operatorname{Ker} \Phi = \tilde{g} \operatorname{Ker} \Phi$$ ✓

hom. property: $\psi((g \operatorname{Ker} \Phi)(\tilde{g} \operatorname{Ker} \Phi)) =$

def. of multiplication in a factor group

Φ homom.

$$= \psi(g\tilde{g} \operatorname{Ker} \Phi) = \Phi(g\tilde{g}) = \Phi(g)\Phi(\tilde{g})$$
$$= \psi(g \operatorname{Ker} \Phi)\, \psi(\tilde{g} \operatorname{Ker} \Phi)$$

Example: Describe the factor group $GL(2,\mathbb{R}) / SL(2,\mathbb{R})$ in terms of a known group.

(i.e. find an isomorphism with a better known group)

Solution: have shown $SL(2,\mathbb{R}) \trianglelefteq GL(2,\mathbb{R})$

Try to find homom. Φ whose kernel $= SL(2,\mathbb{R})$

recall:

- $SL(2,\mathbb{R}) = \{A \in GL(2,\mathbb{R}),\ \det(A) = 1\}$
- have shown: the map $\Phi: GL(2,\mathbb{R}) \to \mathbb{R}^* = \mathbb{R} \setminus \{0\}$

$$A \mapsto \det(A)$$

is a hom.

$\ker \Phi = \{A,\ \Phi(A) = 1\} = \{A,\ \det(A) = 1\} = SL(2,\mathbb{R})$

identity elem. of $\mathbb{R}^*$

can apply First Isom. Theorem:

$$GL(2,\mathbb{R}) / SL(2,\mathbb{R}) \cong \Phi(GL(2,\mathbb{R})) \subset \mathbb{R}^*$$

<u>claim:</u> $\Phi(GL(2,\mathbb{R})) = \mathbb{R}^*$

proof let $x \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$

need to find $A \in GL(2,\mathbb{R})$ with $\det(A) = x$

e.g. $A = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$

✓

<u>Solution:</u> $GL(2,\mathbb{R}) / SL(2,\mathbb{R}) \cong \mathbb{R}^*$